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STOCHASTIC SYSTEMS

Method of the Marginal Asymptotic-Diffusion Analysis for Multiclass Retrial Queue $M_n/GI_n/1$

A. A. Nazarov[*,a] **and E. A. Fedorova**[*,b]

[*] *Tomsk State University, Tomsk, Russia*
e-mail: [a] *nazarov.tsu@gmail.com,* [b] *ekat_fedorova@mail.ru*

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Abstract—This paper considers a queuing system with repeated calls and heterogeneous customers — a multiclass retrial queue. There are a limited number of arrival Poisson processes and one server. If the server is idle at an arrival moment, a customer starts its service with general distribution service time. If the server is busy, a customer goes to an orbit where it performs a random delay distributed exponentially. For the model study, the method of the marginal asymptotic-diffusion analysis under an equivalent long delay of all classes customers in the orbit. As the result, the asymptotic stationary marginal probability distributions of the number of each class customers in the orbit are obtained.

Keywords: multiclass retrial queueing system, marginal asymptotic-diffusion analysis, long delay

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1. INTRODUCTION

Retrial queueing system (RQ) is a class of queueing systems with repeated calls, which is the most widespread mathematical model in IT-systems. The appearance of such models as a new direction of queuing theory (QT) is associated with the development of telecommunication and information technologies. It has been shown that classical models of queuing theory are not suitable for describing some systems where repeated attempts occur. As an example in telephone systems, if the first attempt to call is unsuccessful, a subscriber tries to call again after some random time. Similarly, in communication networks, if a package is transmitted unsuccessfully (due to errors, collisions, server unavailability), there is some delay before a new attempt of transmission. In [1, 2], there are some examples of retrial queuing models application to call centers, cellular networks, computer networks, distributed computing centers, etc.

The detailed description of RQ models is presented in [1, 3]. Despite of the large number of studies, models with several types of customers are studied extremely rarely. The reason is more difficult analytical study of retrial queueing models then other classes of queueing theory (there are analytical formulas only for the simplest models). Also, the problem dimension grows with increasing of the number of classes, so for RQ with N classes it is necessary to study $N+1$-dimensional random process. In case of matrix methods applying [4, 5], the dimension growth has a power law. The same difficulties arise in numerical methods and simulation [6, 7], but some authors propose certain computational optimization algorithms [8].

Often in real communication networks, information is heterogeneous [9, 10]. Different types of transmitted data or types of computational resource requests need different strategies and characteristics of its service. Thus, the study of heterogeneous (or multiclass) models of queuing theory is an urgent scientific problem.

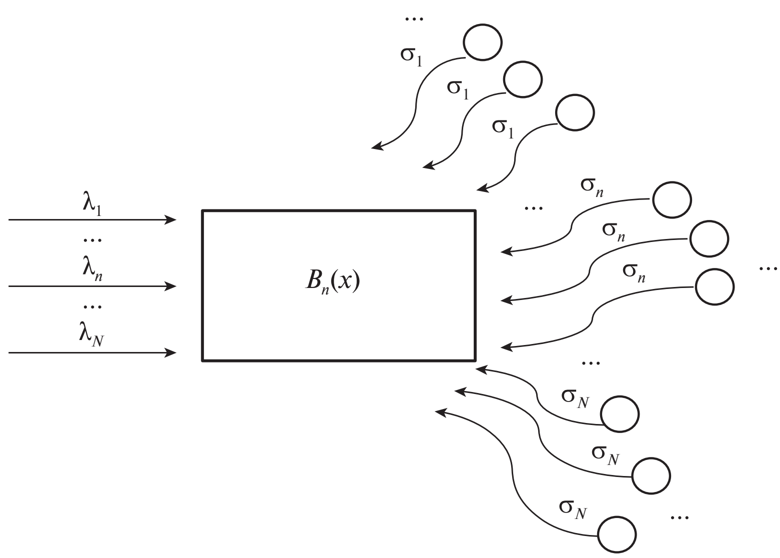
Multiclass RQs are studied by scientific groups supervised by E. Morozov [11–13] and A. Krishnamoorthy [14]. Also RQs with several arrival processes are considered in some papers by B. Kim [15, 16], Y. Shin [17], S. Stepanov [8, 18], etc. Most of the papers [11–13, 15, 16] are devoted to stability analysis of systems (both classical RQ and RQ with constant retrial rate). However, there are no expressions for the probability distributions of the number of customers in the system in this studies, usually only some average characteristics are derived. Of course, mean values and stability analyzing are important in communication networks, but variances and especially types of distributions, gives us a more complete picture. Especially in information systems, where situations of the system overload or loss of information packages are highly undesirable.

In the paper, the original method of the marginal asymptotic-diffusion analysis is proposed for a multiclass retrial queue. This method is the development of the asymptotic-diffusion method outlined in [19, 20] to case of multi-dimensional random processes. The paper is an extension of paper [21] to the case of general service time laws.

2. MATHEMATICAL PROBLEM

Let us consider a multiclass RQ of $M_n/GI_n/1$ type. There are N independent Poisson arrival processes of customers with rates λ_n, where $n = \overline{1, N}$ (so N classes of customers). There is one server. If the server is idle, an arrival nth class customer starts its servicing during random time with cumulative distribution function $B_n(x)$. It is assumed that the given distributions have finite moments of the first and the second orders. If an arrival customer finds the busy server, it goes to the orbit, where it performs a random delay. The delay time is distributed exponentially with rate σ_n for the nth class customer. After the delay, a customer makes a repeated attempt to get the service. If the server is idle the service begins, otherwise the customer returns to the orbit. All customers in the orbit act independently of each other, i.e., there is the multiple random access protocol. Arrival intervals, service times and delays laws of each class customers are mutually independent. The model under consideration is schematically depicted in Figure.

Note that it does not matter to consider one common orbit for all classes of customers (but with different parameters) or several orbits for each class. It is important to distinguish a number of customers of each class in the system at fixed time moment.



Multiclass RQ of $M_n/GI_n/1$ type.

Let $i_n(t)$ be a random processes of the number of the nth class customers in the orbit, $n = \overline{1, N}$, $k(t)$ define states of the server as follows: $k(t) = 0$, if the server is free, $k(t) = n$, if the nth class customer is servicing and $z(t)$ be the remaining servicing time for the current customer on the server.

We denote by $P\{k(t) = 0, i_1(t) = i_1, i_2(t) = i_2, \ldots, i_N(t) = i_N\} = P(0, \mathbf{i}, t)$ and $P\{k(t) = k, i_1(t) = i_1, i_2(t) = i_2, \ldots, i_N(t) = i_N, z(t) < z\} = P(k, \mathbf{i}, z, t)$ the probability that the server has state k and there are $\mathbf{i} = \{i_1, \ldots, i_N\}$ customers in the orbit at time t, and the remaining servicing time is less than z. Process $\{k(t), \mathbf{i}(t), z(t)\}$ is multidimensional continuous-time Markov chain. Let us write the following system of Kolmogorov equations for probability distributions $P(0, \mathbf{i}, t)$ and $P(k, \mathbf{i}, t)$

$$\begin{cases} \dfrac{\partial P(0, \mathbf{i}, t)}{\partial t} = \displaystyle\sum_{n=1}^{N} \dfrac{\partial P(n, \mathbf{i}, 0, t)}{\partial z} - \left(\sum_{n=1}^{N} \lambda_n + \sum_{n=1}^{N} i_n \sigma_n\right) P(0, \mathbf{i}, t), \\ \dfrac{\partial P(k, \mathbf{i}, z, t)}{\partial t} = \dfrac{\partial P(k, \mathbf{i}, z, t)}{\partial z} - \dfrac{\partial P(k, \mathbf{i}, 0, t)}{\partial z} - \displaystyle\sum_{n=1}^{N} \lambda_n P(k, \mathbf{i}, z, t) \\ +\lambda_k P(0, \mathbf{i}, t) B_k(z) + (i_k + 1)\sigma_k P(0, \mathbf{i} + \mathbf{e}_k, t) B_k(z) + \displaystyle\sum_{n=1}^{N} \lambda_n P(k, \mathbf{i} - \mathbf{e}_n, z, t), \end{cases} \tag{1}$$

where $\frac{\partial P(k, \mathbf{i}, 0, t)}{\partial z} = \left.\frac{\partial P(k, \mathbf{i}, z, t)}{\partial z}\right|_{z=0}$, $\mathbf{e}_k$ is vector of $1 \times N$ size with unit kth element and zero others, $k = \overline{1, N}$.

Let us introduce the partial characteristic functions

$$H(0, \mathbf{u}, t) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_N=0}^{\infty} e^{ju_1 i_1} \cdot \ldots \cdot e^{ju_N i_N} P(0, \mathbf{i}, t),$$
$$H(k, \mathbf{u}, z, t) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_N=0}^{\infty} e^{ju_1 i_1} \cdot \ldots \cdot e^{ju_N i_N} P(k, \mathbf{i}, z, t).$$

Then system (1) is rewritten as follows

$$\begin{cases} \dfrac{\partial H(0, \mathbf{u}, t)}{\partial t} = \displaystyle\sum_{n=1}^{N} \dfrac{\partial H(n, \mathbf{u}, 0, t)}{\partial z} - H(0, \mathbf{u}, t) \sum_{n=1}^{N} \lambda_n + \sum_{n=1}^{N} j\sigma_n \dfrac{\partial H(0, \mathbf{u}, t)}{\partial u_n}, \\ \dfrac{\partial H(k, \mathbf{u}, z, t)}{\partial t} = \dfrac{\partial H(k, \mathbf{u}, z, t)}{\partial z} - \dfrac{\partial H(k, \mathbf{u}, 0, t)}{\partial z} + \displaystyle\sum_{n=1}^{N} \lambda_n (e^{ju_n} - 1) H(k, \mathbf{u}, z, t) \\ + \lambda_k H(0, \mathbf{u}, t) B_k(z) - j\sigma_k e^{-ju_k} \frac{\partial H(0, \mathbf{u}, t)}{\partial u_k} B_k(z), \quad k = \overline{1, N}. \end{cases} \tag{2}$$

The direct solving of system (2) is quite difficult, so we propose an original method of the marginal asymptotic-diffusion analysis for its study. This method was developed by the authors and tested for a simpler model in [21], where a fairly high accuracy of the method was shown by comparing with its simulation, and this paper is an extension of research to the case of general service laws.

3. MARGINAL ASYMPTOTIC-DIFFUSION ANALYSIS

The method of the marginal asymptotic-diffusion analysis is a development of the method of asymptotic-diffusion analysis [19] to the case of multidimensional random processes. The considered asymptotic condition is a condition of equivalent long delays of customers in the orbit.

The proposed method contains several stages, which will be presented as subsections:

1) derivation of “marginal” asymptotic equations for one-dimensional process $i_n(t)$ (the number of customers of the “marked” class);

2) deriving of the asymptotic characteristics: means of each class customers numbers and stationary probabilities of the server states;

3) implementation of the asymptotic-diffusion analysis for marked process $i_n(t)$.

Note that the using of the marginal asymptotic method and finding the marginal characteristics of the model for each class does not mean that processes $i_n(t)$, $n = \overline{1,N}$ are independent. This method is a forced measure due to the impossibility of the multidimensional random process studying by known analytical methods, but it allows us to find all characteristics of one-dimensional processes.

3.1. Marginal Asymptotic Equations

The first stage of the proposed method consists in a deriving asymptotic equations for the marginal probability distribution of the number of the marked class customers in the orbit. So let us mark the nth class and study $i_n(t)$ as the marked process.

Any asymptotic method includes an introduction of an infinitesimal value. In the further study, we will suppose that delay rates $\sigma_\nu = \gamma_\nu\sigma$ for $\nu = \overline{1,N}, \nu \neq n$, where $\sigma \to 0$. In this way, there is the asymptotic condition for a long delay condition or a equivalent increasing delay in the orbit for all classes customers.

By denoting $\sigma = \varepsilon$, we introduce the following substitutions

$$u_v = \varepsilon w_v,\ \ \nu = \overline{1,N},\ \ \nu \neq n;\ \ u_n = u;\ \ \mathbf{w}^{(n)} = \{w_1,\ldots,w_{n-1},u,w_{n+1},\ldots,w_N\};$$

$$H(0,\mathbf{u},t) = F(0,\mathbf{w}^{(n)},t), \quad H(k,\mathbf{u},z,t) = F(k,\mathbf{w}^{(n)},z,t).$$

From system (2), we obtain the following equations under $\varepsilon \to 0$

$$\begin{cases} \dfrac{\partial F(0,\mathbf{w}^{(n)},t)}{\partial t} = -F\left(0,\mathbf{w}^{(n)},t\right)\displaystyle\sum_{v=1}^{N}\lambda_\nu + \sum_{\substack{\nu=1\\ \nu\neq n}}^{N} j\gamma_\nu \dfrac{\partial F\left(0,\mathbf{w}^{(n)},t\right)}{\partial w_\nu} \\ + j\sigma_n \dfrac{\partial F\left(0,\mathbf{w}^{(n)},t\right)}{\partial u_n} + \displaystyle\sum_{\nu=1}^{N}\dfrac{\partial F\left(\nu,\mathbf{w}^{(n)},0,t\right)}{\partial z}, \\ \dfrac{\partial F\left(k,\mathbf{w}^{(n)},z,t\right)}{\partial t} = \left(\lambda_n\left(e^{ju_n}-1\right)\right)F\left(k,\mathbf{w}^{(n)},z,t\right) + \dfrac{\partial F\left(k,\mathbf{w}^{(n)},z,t\right)}{\partial z} \\ - \dfrac{\partial F\left(k,\mathbf{w}^{(n)},0,t\right)}{\partial z} + \lambda_k F\left(0,\mathbf{w}^{(n)},t\right)B_k(z) - j\gamma_k \dfrac{\partial F\left(0,\mathbf{w}^{(n)},t\right)}{\partial w_k}B_k(z), \quad k \neq n, \\ \dfrac{\partial F\left(n,\mathbf{w}^{(n)},z,t\right)}{\partial t} = \left(\lambda_n\left(e^{ju_n}-1\right)\right)F\left(n,\mathbf{w}^{(n)},z,t\right) + \dfrac{\partial F(n,\mathbf{w}^{(n)},z,t)}{\partial z} \\ - \dfrac{\partial F\left(n,\mathbf{w}^{(n)},0,t\right)}{\partial z} + \lambda_n F\left(0,\mathbf{w}^{(n)},t\right)B_n(z) - j\sigma_n e^{-ju_n}\dfrac{\partial F\left(0,\mathbf{w}^{(n)},t\right)}{\partial u_n}B_n(z). \end{cases} \tag{3}$$

From the form of equations (3), it can be concluded that the solution of this system can be written as follows

$$F(0,\mathbf{w}^{(n)},t) = H_n(0,u_n,t)\times\exp\left\{\sum_{\nu\neq n} jw_\nu x_\nu\right\},$$
$$F(k,\mathbf{w}^{(n)},z,t) = H_n(k,u_n,z,t)\times\exp\left\{\sum_{\nu\neq n} jw_\nu x_\nu\right\}, \quad (4)$$

where x_n are unknown parameters and functions $H_n(0,u_n,t)$ and $H_n(k,u_n,z,t)$ satisfy the following system of differential equations

$$\begin{cases} \dfrac{\partial H_n(0,u_n,t)}{\partial t} = -H_n(0,u_n,t)\left(\sum\limits_{\nu=1}^{N}\lambda_\nu + \sum\limits_{\substack{\nu=1\\ v\neq n}}^{N}\gamma_\nu x_\nu\right) \\ +\, j\sigma_n\dfrac{\partial H_n(0,u_n,t)}{\partial u_n} + \sum\limits_{\nu=1}^{N}\dfrac{\partial H_n(\nu,u_n,0,t)}{\partial z}, \\ \dfrac{\partial H_n(k,u_n,z,t)}{\partial t} = \lambda_n(e^{ju_n}-1)H_n(k,u_n,z,t) + \dfrac{\partial H_n(k,u_n,z,t)}{\partial z} \\ -\, \dfrac{\partial H_n(k,u_n,0,t)}{\partial z} + H_n(0,u_n,t)(\lambda_k+\gamma_k x_k)B_k(z),\ \ k\neq n, \\ \dfrac{\partial H_n(n,u_n,z,t)}{\partial t} = \lambda_n(e^{ju_n}-1)H_n(n,u_n,z,t) + \dfrac{\partial H_n(n,u_n,z,t)}{\partial z} \\ -\, \dfrac{\partial H_n(\nu,u_n,0,t)}{\partial z} + \lambda_n H_n(0,u_n,t)B_n(z) - j\sigma_n e^{-ju_n}\dfrac{\partial H_n(0,u_n,t)}{\partial u_n}B_n(z). \end{cases} \quad (5)$$

Unknown functions $H_n(k,u_n,z,t)$ have the meaning of the asymptotic partial characteristic functions of the number of customers of the marked class in the orbit. By solving system (5), we can obtain the marginal distributions for each class of customers numbers. However, these equations contain unknown parameters x_k, which defines the asymptotic means. In the next paragraph, we will derive expressions for them.

3.2. Asymptotic Means

For finding of parameters x_k, $k=\overline{1,N}$, let us return to system (2) and write it under the steady state.

$$\begin{cases} \sum\limits_{n=1}^{N}\dfrac{\partial H(n,\mathbf{u},0)}{\partial z} - H(0,\mathbf{u})\sum\limits_{n=1}^{N}\lambda_n + \sum\limits_{n=1}^{N} j\sigma_n\dfrac{\partial H(0,\mathbf{u})}{\partial u_n} = 0, \\ \dfrac{\partial H(k,\mathbf{u},z)}{\partial z} - \dfrac{\partial H(k,\mathbf{u},0)}{\partial z} + \sum\limits_{n=1}^{N}\lambda_n(e^{ju_n}-1)H(k,\mathbf{u},z) \\ +\lambda_k H(0,\mathbf{u})B_k(z) - j\sigma_k e^{-ju_k}\dfrac{\partial H(0,\mathbf{u})}{\partial u_k}B_k(z) = 0,\ \ k=\overline{1,N}. \end{cases} \quad (6)$$

Summarizing equations for all $k=\overline{0,N}$ and taking $z\to\infty$, we obtain the following additional equation called as a consistent equation

$$\sum_{n=1}^{N}(e^{ju_n}-1)\left(\lambda_n\sum_{k=1}^{N}H(k,\mathbf{u}) + j\sigma_k e^{-ju_k}\frac{\partial H(0,\mathbf{u})}{\partial u_k}\right) = 0. \quad (7)$$

Let us introduce the asymptotic substitutions in (6) and (7)

$$\sigma_n = \gamma_n \sigma, \ \ \sigma = \varepsilon, \ \ u_n = \varepsilon w_n, \ \ H(0, \mathbf{u}) = F(0, \mathbf{w}, \varepsilon), \ \ H(k, \mathbf{u}, z) = F(k, \mathbf{w}, z, \varepsilon).$$

Thus we obtain the following system

$$\begin{cases} \sum\limits_{n=1}^{N} \dfrac{\partial F(n, \mathbf{w}, 0, \varepsilon)}{\partial z} - F(0, \mathbf{w}, \varepsilon) \sum\limits_{n=1}^{N} \lambda_n + \sum\limits_{n=1}^{N} j\gamma_n \dfrac{\partial F(0, \mathbf{w}, \varepsilon)}{\partial w_n} = O(\varepsilon), \\ \dfrac{\partial F(k, \mathbf{w}, z, \varepsilon)}{\partial z} - \dfrac{\partial F(k, \mathbf{w}, 0, \varepsilon)}{\partial z} + \sum\limits_{n=1}^{N} \lambda_n (e^{j\varepsilon w_n} - 1) F(k, \mathbf{w}, z, \varepsilon) \\ + \lambda_k F(0, \mathbf{w}, \varepsilon) B_k(z) - j\gamma_k e^{-j\varepsilon w_k} \dfrac{\partial F(0, \mathbf{w}, \varepsilon)}{\partial w_k} B_k(z) = O(\varepsilon), \ \ k = \overline{1, N}, \\ \sum\limits_{n=1}^{N} (e^{j\varepsilon w_n} - 1) \left(\lambda_n \sum\limits_{k=1}^{N} F(k, \mathbf{w}, z, \varepsilon) + j\gamma_k e^{-j\varepsilon w_k} \dfrac{\partial F(0, \mathbf{w}, \varepsilon)}{\partial w_k} \right) = O(\varepsilon). \end{cases} \tag{8}$$

After some mathematical transformations under $\varepsilon \to 0$, we have

$$\begin{cases} \sum\limits_{n=1}^{N} \dfrac{\partial F(n, \mathbf{w}, 0)}{\partial z} - F(0, \mathbf{w}) \sum\limits_{n=1}^{N} \lambda_n + \sum\limits_{n=1}^{N} j\gamma_n \dfrac{\partial F(0, \mathbf{w})}{\partial w_n} = 0, \\ \dfrac{\partial F(k, \mathbf{w}, z)}{\partial z} - \dfrac{\partial F(k, \mathbf{w}, 0)}{\partial z} + \lambda_k F(0, \mathbf{w}) B_k(z) - j\gamma_k \dfrac{\partial F(0, \mathbf{w})}{\partial w_k} B_k(z) = 0, \ \ k = \overline{1, N}, \\ \sum\limits_{n=1}^{N} j w_n \left(\lambda_n \sum\limits_{k=1}^{N} F(k, \mathbf{w}, z) + j\gamma_k \dfrac{\partial F(0, \mathbf{w})}{\partial w_k} \right) = 0. \end{cases} \tag{9}$$

The solution of the system above has the form

$$F(0, \mathbf{w}) = r_0 \times \exp\left\{ \sum_{n=1}^{N} j w_n x_n \right\}, \ \ F(k, \mathbf{w}, z) = r_k(z) \times \exp\left\{ \sum_{n=1}^{N} j w_n x_n \right\}.$$

From system (9), we obtain that

$$\begin{cases} \sum\limits_{n=1}^{N} r_n'(0) - r_0 \sum\limits_{n=1}^{N} (\lambda_n + \gamma_n x_n) = 0, \\ r_k'(z) - r_k'(0) + r_0 B_k(z)(\lambda_k + \gamma_k x_k) = 0, \ k = \overline{1, N}, \\ \lambda_n \sum\limits_{k=1}^{N} r_k(z) - \gamma_n x_n r_0 = 0, \ n = \overline{1, N}, \end{cases} \tag{10}$$

where $r_n'(0) = \frac{dr_n(z)}{dz}\Big|_{z=0}$.

Let us denote $\kappa_n = \lambda_n + \gamma_n x_n$. Then the following expressions are derived from system (10)

$$r_k(z) = r_0 \kappa_k \int_0^z (1 - B_k(x)) dx, \quad r_k = \lambda_k b_k^{(1)}, \tag{11}$$

where $\kappa_k = \lambda_k / r_0$, $k = \overline{1, N}$, $b_k^{(1)} = \int_0^\infty (1 - B_k(x)) dx$ and r_0 is defined from the normalization condition as

$$r_0 = 1 - \sum_{k=1}^{N} r_k \ \text{ or } \ r_0 = \left(1 + \sum_{k=1}^{N} \kappa_k b_k^{(1)} \right)^{-1}. \tag{12}$$

So the asymptotic means of each class customers numbers are $m_k = (\kappa_k - \lambda_k)$, where $\kappa_k = \lambda_k / r_0$.

3.3. Asymptotic-Diffusion Analysis

Let us return to equations for the marked process (5). We write the consistent equation by summing all equations for $z \to \infty$.

$$\frac{\partial H_n(u_n,t)}{\partial t} = (e^{ju_n}-1)\left(\lambda_n \sum_{k=1}^{N} H_u(k,u_n,t) + j\sigma_n e^{-ju_n}\frac{\partial H_n(0,u_n,t)}{\partial u_n}\right). \tag{13}$$

3.3.1. First Asymptotics. Let us denote

$$\sigma_n = \varepsilon,\ \ \sigma_n t = \varepsilon t = \tau,\ \ u_n = \varepsilon w,$$

$$H_n(0,u_n,t) = F_n(0,w,\tau,\varepsilon), \quad H_n(k,u_n,z,t) = F_n(k,w,\tau,z,\varepsilon).$$

We substitute the notations into equations (5).

$$\begin{cases} \varepsilon\dfrac{\partial F_n(0,w,\tau,\varepsilon)}{\partial \tau} = -F_n(0,w,\tau,\varepsilon)\left(\lambda_n + \sum\limits_{\substack{v=1\\ v\neq n}}^{N}\kappa_v\right) \\ \quad + j\dfrac{\partial F_n(0,w,\tau,\varepsilon)}{\partial w} + \sum\limits_{v=1}^{N}\dfrac{\partial F_n(v,w,\tau,0,\varepsilon)}{\partial z}, \\ \varepsilon\dfrac{\partial F_n(k,w,\tau,z,\varepsilon)}{\partial \tau} = \lambda_n(e^{j\varepsilon w}-1)F_n(k,w,\tau,\varepsilon) + \dfrac{\partial F_n(k,w,\tau,z,\varepsilon)}{\partial z} \\ \quad - \dfrac{\partial F_n(k,w,\tau,0,\varepsilon)}{\partial z} + F_n(0,w,\tau,\varepsilon)\kappa_k B_k(z),\ \ k\neq n, \\ \varepsilon\dfrac{\partial F_n(n,w,\tau,z,\varepsilon)}{\partial \tau} = \lambda_n(e^{j\varepsilon w}-1)F_n(n,w,\tau,z,\varepsilon) + \dfrac{\partial F_n(n,w,\tau,z,\varepsilon)}{\partial z} \\ \quad - \dfrac{\partial F_n(n,w,\tau,0,\varepsilon)}{\partial z} + \lambda_n F_n(0,w,\tau,\varepsilon)B_n(z) - je^{-j\varepsilon w}\dfrac{\partial F_n(0,w,\tau,\varepsilon)}{\partial w}B_n(z). \end{cases} \tag{14}$$

From equation (13), we have:

$$\varepsilon\sum_{k=0}^{N}\frac{\partial F_n(k,w,\tau,\varepsilon)}{\partial \tau} = (e^{j\varepsilon w}-1)\left(\lambda_n\sum_{k=1}^{N}F_n(k,w,\tau,\varepsilon) + je^{-j\varepsilon w}\frac{\partial F_n(0,w,\tau,\varepsilon)}{\partial w}\right). \tag{15}$$

Under limit condition $\varepsilon \to 0$, equations (14) are written as

$$\begin{cases} -F_n(0,w,\tau)\left(\lambda_n + \sum\limits_{\substack{v=1\\ v\neq n}}^{N}\kappa_v\right) + j\,\dfrac{\partial F_n(0,w,\tau)}{\partial w} + \sum\limits_{v=1}^{N}\dfrac{\partial F_n(v,w,0,\tau)}{\partial z} = 0, \\ \dfrac{\partial F_n(k,w,z,\tau)}{\partial z} - \dfrac{\partial F_n(k,w,0,\tau)}{\partial z} + F_n(0,w,\tau)\kappa_k B_k(z) = 0,\ \ k\neq n, \\ \dfrac{\partial F_n(n,w,z,\tau)}{\partial z} - \dfrac{\partial F_n(n,w,0,\tau)}{\partial z} + \lambda_n F_n(0,w,\tau)B_n(z) - j\dfrac{\partial F_n(0,w,\tau)}{\partial w}B_n(z) = 0. \end{cases} \tag{16}$$

The form of equations (16) lets us make the conclusion that functions $F_n(0,w,\tau)$ and $F_n(k,w,z,\tau)$ can be written as

$$\begin{aligned} F_n(0,w,\tau) &= R_0(x)e^{jw\cdot x(\tau)}, \\ F_n(k,w,z,\tau) &= R_k(x,z)e^{jw\cdot x(\tau)}, \end{aligned} \tag{17}$$

where index n (the number of the marked class) is missed in notations $R_0(x)$ and $R_k(x,z)$, however, we need keep in the mind that the expressions for these functions will differ for each class. Further, to simplify the expressions, we will write x instead of $x(\tau)$.

By substituting (17) into (16), we obtain

$$\begin{cases} \sum\limits_{v=1}^{N} \dfrac{\partial R_v(x,0)}{\partial z} = R_0(x)\left(\lambda_n + \sum\limits_{\substack{v=1 \\ v\neq n}}^{N} \kappa_v + x\right), \\ \dfrac{\partial R_k(x,z)}{\partial z} = \dfrac{\partial R_k(x,0)}{\partial z} - R_0(x)\kappa_k B_k(z) = 0,\ k \neq n, \\ \dfrac{\partial R_n(x,z)}{\partial z} = \dfrac{\partial R_n(x,0)}{\partial z} - R_0(x)B_k(z)(\lambda_n + x). \end{cases} \tag{18}$$

It is easy to derive that

$$\begin{aligned} R_k(x,z) &= R_0(x)\kappa_k \int\limits_0^z (1 - B_k(x))dx,\ k \neq n, \\ R_n(x,z) &= R_0(x)(\lambda_n + x) \int\limits_0^z (1 - B_n(x))dx. \end{aligned} \tag{19}$$

For $z \to \infty$, we obtain

$$\begin{aligned} R_k(x) &= R_0(x)\kappa_k b_k^{(1)}, \quad k \neq n, \\ R_n(x) &= R_0(x)(\lambda_n + x)b_n^{(1)}, \end{aligned} \tag{20}$$

where $R_0(x)$ is defined by normalization condition

$$R_0(x) = \left(1 + (\lambda_n + x)b_n^{(1)} + \sum_{\substack{k=1 \\ v\neq n}}^{N} \kappa_k b_k^{(1)}\right)^{-1}. \tag{21}$$

Let us return to equation (15). We perform some transformations and write equations under $\varepsilon \to 0$.

$$\sum_{k=0}^{N} \frac{\partial F_n(k,w,\tau)}{\partial \tau} = jw\left(j\frac{\partial F_n(0,w,\tau)}{\partial w} + \lambda_n \sum_{k=1}^{N} F_n(k,w,\tau)\right).$$

Substituting (17), we finally obtain that the asymptotic mean number of customers of the nth class in the orbit $x(\tau)$ is determined by the equation

$$\frac{dx(\tau)}{d\tau} = a(x(\tau)),$$

where

$$a(x) = \lambda_n(1 - R_0(x)) - R_0(x)x \tag{22}$$

has the meaning of the drift coefficient of the random process under study.

3.3.2. Second Asymptotics. In system (5), we make substitutions

$$H_n(0,u_n,t) = H_n^{(2)}(0,u_n,t)\exp\left\{\frac{ju_n}{\sigma_n}x(\sigma_n t)\right\},$$

$$H_n(k,u_n,z,t) = H_n^{(2)}(k,u_n,z,t)\exp\left\{\frac{ju_n}{\sigma_n}x(\sigma_n t)\right\}.$$

Further, to simplify the expressions, we will write x instead of $x(\sigma_n t)$.

So we derive the following system of equations

$$\begin{cases} \dfrac{\partial H_n^{(2)}(0,u_n,t)}{\partial t} + ju_n a(x) H_n^{(2)}(0,u_n,t) \\ = -H_n^{(2)}(0,u_n,t)\left(\lambda_n + \sum\limits_{\substack{v=1 \\ v\neq n}}^{N}\kappa_v\right) + j\sigma_n \dfrac{\partial H_n^{(2)}(0,u_n,t)}{\partial u_n} \\ -xH_n^{(2)}(0,u_n,t) + \sum\limits_{k=1}^{N}\dfrac{\partial H_n^{(2)}(k,u_n,0,t)}{\partial z}, \\ \\ \dfrac{\partial H_n^{(2)}(k,u_n,z,t)}{\partial t} + ju_n a(x) H_n^{(2)}(k,u_n,z,t) \\ = \lambda_n(e^{ju_n}-1)H_n^{(2)}(k,u_n,z,t) + \dfrac{\partial H_n^{(2)}(k,u_n,z,t)}{\partial z} \\ -\dfrac{\partial H_n^{(2)}(k,u_n,0,t)}{\partial z} + H_n^{(2)}(0,u_n,t)B_k(z)\kappa_k,\ k\neq n, \\ \\ \dfrac{\partial H_n^{(2)}(n,u_n,z,t)}{\partial t} + ju_n a(x) H_n^{(2)}(n,u_n,z,t) \\ = \lambda_n(e^{ju_n}-1)H_n^{(2)}(n,u_n,z,t) + \dfrac{\partial H_n^{(2)}(n,u_n,z,t)}{\partial z} \\ -\dfrac{\partial H_n^{(2)}(n,u_n,0,t)}{\partial z} + \lambda_n H_n^{(2)}(0,u_n,t)B_n(z) \\ \\ -j\sigma_n e^{-ju_n}\dfrac{\partial H_n^{(2)}(0,u_n,t)}{\partial u_n}B_n(z) + e^{-ju_n}H_n^{(2)}(0,u_n,t)xB_n(z). \end{cases} \tag{23}$$

From consistent equation (6), we have an additional equation

$$\frac{\partial H_n^{(2)}(u_n,t)}{\partial t} + ju_n a(x) H_n^{(2)}(u_n,t)$$
$$= (e^{ju_n}-1)\left(\lambda_n\sum_{k=1}^{N}H_n^{(2)}(k,u_n,t) + j\sigma_n e^{-ju_n}\frac{\partial H_n^{(2)}(0,u_n,t)}{\partial u_n} - e^{-ju_n}H_n^{(2)}(0,u_n,t)x\right). \tag{24}$$

Let us denote

$$\sigma_n = \varepsilon^2,\ \sigma_n t = \varepsilon^2 t = \tau,\ u_n = \varepsilon w,$$

$$H_n^{(2)}(0,u_n,t) = F_n^{(2)}(0,w,\tau,\varepsilon),\ H_n^{(2)}(k,u_n,z,t) = F_n^{(2)}(k,w,z,\tau,\varepsilon).$$

From equation (23), after some transformations we obtain the following asymptotic equations

$$\begin{cases} \varepsilon^2 \dfrac{\partial F_n^{(2)}(0,w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_n^{(2)}(0,w,\tau,\varepsilon) \\ = -F_n^{(2)}(0,w,\tau,\varepsilon)\left(\lambda_n + \sum\limits_{\substack{v=1 \\ v\neq n}}^{N} \kappa_v\right) + j\varepsilon \dfrac{\partial F_n^{(2)}(0,w,\tau,\varepsilon)}{\partial w} \\ - xF_n^{(2)}(0,w,\tau,\varepsilon) + \sum\limits_{k=1}^{N} \dfrac{\partial F_n^{(2)}(k,w,0,\tau,\varepsilon)}{\partial z} + O(\varepsilon^2), \\ \varepsilon^2 \dfrac{\partial F_n^{(2)}(k,w,\tau,z,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_n^{(2)}(k,w,\tau,z,\varepsilon) \\ = \lambda_n \left(j\varepsilon w + \dfrac{(j\varepsilon w)^2}{2}\right) F_n^{(2)}(n,w,z,\tau,\varepsilon) + F_n^{(2)}(0,w,\tau,\varepsilon) B_k(z) \kappa_k \\ + \dfrac{\partial F_n^{(2)}(k,w,z,\tau,\varepsilon)}{\partial z} - \dfrac{\partial F_n^{(2)}(k,w,0,\tau,\varepsilon)}{\partial z} + O(\varepsilon^2), \quad k \neq n, \\ \varepsilon^2 \dfrac{\partial F_n^{(2)}(n,w,\tau,z,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_n^{(2)}(n,w,\tau,z,\varepsilon) \\ = \lambda_n \left(j\varepsilon w + \dfrac{(j\varepsilon w)^2}{2}\right) F_n^{(2)}(n,w,z,\tau,\varepsilon) \\ + \lambda_n F_n^{(2)}(0,w,\tau,\varepsilon) B_n(z) + \dfrac{\partial F_n^{(2)}(n,w,z,\tau,\varepsilon)}{\partial z} \\ - \dfrac{\partial F_n^{(2)}(n,w,0,\tau,\varepsilon)}{\partial z} - j\varepsilon(1 - j\varepsilon w) \dfrac{\partial F_n^{(2)}(0,w,\tau,\varepsilon)}{\partial w} B_n(z) \\ + \left(1 - j\varepsilon w + \dfrac{(j\varepsilon w)^2}{2}\right) F_n^{(2)}(0,w,\tau,\varepsilon) x B_n(z) + O(\varepsilon^2). \end{cases} \tag{25}$$

From equation (24), we have

$$\varepsilon^2 \frac{\partial F_n^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x) F_n^{(2)}(w,\tau,\varepsilon) = \left(j\varepsilon w + \frac{(j\varepsilon w)^2}{2}\right) \times \left(\lambda_n \sum_{k=1}^{N} F_n^{(2)}(k,w,\tau,\varepsilon) + j\varepsilon(1 - j\varepsilon w) \frac{\partial F_n^{(2)}(0,w,\tau,\varepsilon)}{\partial w} - x\left(1 - j\varepsilon w + \frac{(j\varepsilon w)^2}{2}\right) F_n^{(2)}(0,w,\tau,\varepsilon)\right). \tag{26}$$

Let us find the solution of system (25)–(26) in the following form

$$\begin{aligned} F_n^{(2)}(0,w,\tau,\varepsilon) &= \Phi(w,\tau)(R_0(x) + jw\varepsilon f_0(x)) + O(\varepsilon^2), \\ F_n^{(2)}(k,w,z,\tau,\varepsilon) &= \Phi(w,\tau)(R_k(x,z) + jw\varepsilon f_k(x,z)) + O(\varepsilon^2). \end{aligned} \tag{27}$$

Substituting the expressions above in system (25)–(26) and taking into account (16) and (18), we obtain the following equations after some transformations and under $\varepsilon \to 0$

$$\begin{cases} -f_0(x)\left(\lambda_n + \sum\limits_{\substack{v=1\\v\neq n}}^{N}\kappa_v + x\right) + \sum\limits_{k=1}^{N}\dfrac{\partial f_k(x,0)}{\partial z} = a(x)R_0(x) - R_0(x)\dfrac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)},\\ \dfrac{\partial f_k(x,z)}{\partial z} - \dfrac{\partial f_k(x,0)}{\partial z} + f_0(x)\kappa_k B_k(z) = R_k(x,z)(a(x)-\lambda_n), \quad k\neq n,\\ f_0(x)(\lambda_n + x) + \dfrac{\partial f_n(x,z)}{\partial z} - \dfrac{\partial f_n(x,0)}{\partial z}\\ = R_n(x,z)(a(x)-\lambda_n) + xR_0(x)B_n(z) + R_0(x)\dfrac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}B_n(z). \end{cases} \tag{28}$$

Comparing equations (28) with (18) and using the principle of superposition, we conclude that the solution of (28) can be written as

$$\begin{aligned} f_0(x) &= CR_0(x) + g_0(x) - \phi_0(x)\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)},\\ f_k(x,z) &= CR_k(x,z) + g_k(x,z) - \phi_k(x,z)\frac{\partial\Phi(w,\tau)/\partial w}{w\Phi(w,\tau)}, \end{aligned} \tag{29}$$

where C is normalizing constant and functions $g_k(x,z)$ and $\phi_k(x,z)$ are defined by the following equations systems

$$\begin{cases} -\phi_0(x)\left(\lambda_n + \sum\limits_{\substack{v=1\\v\neq n}}^{N}\kappa_v + x\right) + \sum\limits_{k=1}^{N}\dfrac{\partial \phi_k(x,0)}{\partial z} = R_0(x),\\ \dfrac{\partial \phi_k(x,z)}{\partial z} - \dfrac{\partial \phi_k(x,0)}{\partial z} + \phi_0(x)\kappa_k B_k(z) = 0, \ k\neq n,\\ \dfrac{\partial \phi_n(x,z)}{\partial z} - \dfrac{\partial \phi_n(x,0)}{\partial z} + (\lambda_n + x)\phi_0(x)B_n(z) = -R_0(x)B_n(z), \end{cases} \tag{30}$$

$$\begin{cases} -g_0(x)\left(\lambda_n + \sum\limits_{\substack{v=1\\v\neq n}}^{N}+\kappa_v + x\right) + \sum\limits_{k=1}^{N}\dfrac{\partial g_k(x,0)}{\partial z} = a(x)R_0(x),\\ \dfrac{\partial g_k(x,z)}{\partial z} - \dfrac{\partial g_k(x,0)}{\partial z} + g_0(x)\kappa_k B_k(z) = R_k(x,z)(a(x)-\lambda_n), \ k\neq n,\\ \dfrac{\partial g_n(x,z)}{\partial z} - \dfrac{\partial g_n(x,0)}{\partial z} + g_0(x)(\lambda_n + x)B_n(z) = R_n(x,z)(a(x)-\lambda_n) + xR_0(x)B_n(z). \end{cases} \tag{31}$$

For uniqueness of the solutions, we supplement the systems with conditions $\sum\limits_{k=1}^{N}\phi_k(x) = -\phi_0(x)$ and $\sum\limits_{k=1}^{N} g_k(x) = -g_0(x)$. Then it can be written that

$$\phi_n(x,z) = \phi_0(x)\int\limits_0^z(\lambda_n(1-B_n(y)-x))dy, \quad \phi_k(x,z) = \phi_0(x)\kappa_k\int\limits_0^z(1-B_k(y))dy, \ \ k\neq n,$$

where under $z\to\infty$

$$\phi_k(x) = R_k'(x). \tag{32}$$

In the same way from the system (31), we obtain:

$$g_k(x,z) = g_0(x)\kappa_k \int_0^z (1 - B_k(y))dy + (a(x) - \lambda_n) \int_0^z (R_k(x,y) - R_k(x))dy,\ k \neq n,$$

$$g_n(x,z) = (\lambda_n + x)g_0(x) \int_0^z (\lambda_n(1 - B_n(y))dy$$

$$+ (a(x) - \lambda_n) \int_0^z (R_n(x,y) - R_n(x))dy - xR_0(x) \int_0^z ((1 - B_n(y))dy,$$

where

$$g_0(x) = R_0(x) \frac{(\lambda_n - a(x))((\lambda_n + x)b_n^{(2)} + 2xb_n^{(1)} + (\lambda_n - a(x)) \sum\limits_{\substack{k=1 \\ v\neq n}}^{N} \kappa_k b_k^{(2)}}{2\left(1 + (\lambda_n + x)b_n^{(1)} + \sum\limits_{\substack{k=1 \\ v\neq n}}^{N} \kappa_k b_k^{(1)}\right)}, \quad (33)$$

and $b_k^{(1)} = \int_0^\infty x dB_k(x)$, $b_k^{(2)} = \int_0^\infty x^2 dB_k(x)$.

The last step of the asymptotic analysis is finding of function $\Phi(w,\tau)$. For this, we substitute substitutions (27) into equation (26). After some transformations, we write the resulting equation under $\varepsilon \to 0$.

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = \frac{(jw)^2}{2} \Phi(w,\tau)$$
$$\times \left(a(x) + 2x(R_0(x) - g_0(x)) - 2\lambda_n g_0(x) + (2\phi_0(x)(x + \lambda_n) + 2R_0(x)) \frac{\partial \Phi(w,\tau)}{\partial w} \frac{1}{w\Phi(w,\tau)} \right).$$

In this way, we derive that $\Phi(w,\tau)$ defines by the equation

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = w \frac{\partial \Phi(w,\tau)}{\partial w} a'(x) + \frac{(jw)^2}{2} \Phi(w,\tau) b(x), \quad (34)$$

where

$$b(x) = a(x) + 2x(R_0(x) - g_0(x)) - 2\lambda_n g_0(x). \quad (35)$$

Using the inverse Fourier transform to (34) and denoting $P(y,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-jwy} \Phi(w,\tau) dw$, we obtain that

$$\frac{\partial P(y,\tau)}{\partial \tau} = -\frac{\partial}{\partial y} \left(P(y,\tau) y a'(x) \right) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (P(y,\tau) b(x)),$$

which is Fokker–Planck equation for the probability distribution density $P(y,\tau)$ of diffusion process $y(\tau)$, which is the solution of the stochastic differential equation

$$dy(\tau) = y(\tau) a^*(x) d\tau + \sqrt{b(x)} dw(\tau).$$

3.4. Result of Asymptotic-Diffusion Analysis

Combining the results of both asymptotics, we introduce process $z(\tau) = x(\tau) + \sqrt{\sigma_n} \times y(\tau)$, which is the solution of the stochastic equation

$$dz(\tau) = a(z) + \sqrt{\sigma_n b(z)} dw(\tau),$$

where $w(\tau)$ is Wiener process.

The probability distribution density $P(z,\tau)$ of diffusion process $z(\tau)$ satisfies the Fokker–Planck equation

$$\frac{\partial P(z,\tau)}{\partial \tau} = -\frac{\partial}{\partial z}\left(P(z,\tau)a(z)\right) + \frac{1}{2}\frac{\partial^2}{\partial z^2}(P(z,\tau)\sigma_n b(z)).$$

Then the stationary probabilities of process $z(\tau)$ are expressed as

$$P(z) = \frac{C}{b(z)}\exp\left(\frac{\sigma_n}{2}\int\limits_0^z \frac{a(x)}{b(x)}dx\right).$$

Returning to the study purpose, the result of the method of marginal asymptotic-diffusion analysis is approximation of the probability distribution of process $i_n(t)$ (the number of customers of the nth class in the orbit) in the following form

$$P(i_n) \approx \frac{C}{b(\sigma_n i_n)}\exp\left(\frac{\sigma_n}{2}\int\limits_0^{\sigma_n i_n} \frac{a(x)}{b(x)}dx\right), \qquad (36)$$

where $C = \text{const}$ is calculated from normalization condition $\sum_{i_n=0}^{\infty} P(i_n) = 1$.

Note that formula (36) can be used for all classes marginal distributions, but it is worth remembering that parameters $a(x)$, $b(x)$ in (22) and (35), as $R_k(x)$, $g_k(x)$, $\phi_k(x)$ in (20) and (32)–(33) will be different for each class.

Thus, the asymptotic marginal probability distribution of the number of the marked class customers is dervied, that allows us to find all necessary characteristics of the system under the consideration, for example, means of the number of customers of each class in the orbit $m_n = \kappa_n - \lambda_n$, the probability of the server downtime r_0 according to formula (12), moments and quantile of the desirable order, etc.

4. CONCLUSION

In the paper, a multiclass RQ system of $M_n/GI_n/1$ type is studied. The original method of the marginal asymptotic diffusion analysis is proposed. As the result, the expression for the approximation of the probability distribution of the number of the marked class customers in the orbit is obtained, that makes possible to assess any characteristics of the system. Note that the study does not establish any criteria to mark the class, thus, the obtained formulas can be used to calculate the marginal probability distribution of the number of each class customers. In future research, the asymptotic results can be applied to modeling and optimizing real telecommunication networks.

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